Two-Sided Linear Quaternionic Equations: History, Structure, and the Discriminant Space Method

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Outline

Hamilton and the Birth of Quaternions (1843)

The 16th century Italian mathematician Gerolamo Cardano (1501–1576) is credited with introducing complex numbers in his attempts to find solutions to cubic equations.

W. R. Hamilton (1805-1865) discovered quaternions in 1843 while he was searching for a way to construct a number system bigger than complex number. First he was trying to multiply vectors in $\mathbb{R}^3$. He failed. Six years later he was successful!

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

Relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k.$$

First noncommutative division algebra.

Hamilton (1843), Proc. Royal Irish Academy.

Structural Importance of Quaternions

Key properties:

- $\mathbb{H}$ is a 4-dimensional $\mathbb{R}$-vector space.
- $\mathbb{H}$ is a division algebra.
- Center: $\mathbb{R}$.
- Noncommutative but associative.

Frobenius theorem:

Only finite-dimensional associative real division algebras are:

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}.$$

The quaternion algebra plays an important role in many areas, such as computer science, quantum mechanics, signal processing, color image processing, and other fields. In recent years, renewed interest in the zeros of quaternionic polynomials has led to further developments of quaternionic analytical tools.

Recall that if $(R, +, \cdot)$ is a ring and if $\varnothing \neq S \subseteq R$, then S is a subring of R $(S \leq R)$ iff $a - b, ab \in S, \forall a, b \in S$.
We know that $(M_{2\times 2}(\mathbb{C}), +, \cdot)$ is a noncommutative unital ring. Now, in this example, $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the elements

$$\vec{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

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Then, define a set

$$\mathbb{H} = \left\{ a \cdot 1 + b\vec{i} + c\vec{j} + d\vec{k} | a, b, c, d \in \mathbb{R} \right\}.$$

Then $\mathbb{H}$ is actually the following 4 dimensional vector space over $\mathbb{R}$ and

$$\mathbb{H} = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix} \middle| z_1, z_2 \in \mathbb{C} \right\} \subseteq M_{2\times 2}(\mathbb{C}),$$

where $\overline{z}_1$ is the conjugate of the complex number z_1.
But $\mathbb{H}$ does not seem a vector space over $\mathbb{C}$. It is. Actually, $\mathbb{H}$ can be expressed as a linear combination over $\mathbb{R}$, or a linear combination over $\mathbb{C}$ shown below:

$$\mathbb{H} = \mathbb{R}\cdot 1 + \mathbb{R}\cdot\vec{i} + \mathbb{R}\cdot\vec{j} + \mathbb{R}\cdot\vec{k} = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\vec{j}.$$

Actually,

$$a + b\vec{i} + c\vec{j} + d\vec{k} = \begin{pmatrix} a+b\vec{i} & c+d\vec{i} \\ -c+d\vec{i} & a-b\vec{i} \end{pmatrix}.$$

Theorem

$\mathbb{H} \leq M_{2\times 2}(\mathbb{C})$.

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$$\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -\overline{z}_4 & \overline{z}_3 \end{pmatrix} = \begin{pmatrix} z_1 z_3 - z_2 \overline{z}_4 & z_1 z_4 + z_2 \overline{z}_3 \\ -\overline{z}_2 z_3 - \overline{z}_1 \overline{z}_4 & -\overline{z}_2 z_4 + \overline{z}_1 \overline{z}_3 \end{pmatrix} \in \mathbb{H}.$$

So $\mathbb{H} \leq M_{2\times 2}(\mathbb{C})$. □

By direct computations we have

$$\vec{i}^2 = -1, \vec{j}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1,$$

$$\vec{k}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1,$$

and

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \vec{j}\vec{k} = \vec{i} = -\vec{k}\vec{j}, \vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

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Like with complex numbers, we can define the **conjugate** of $z = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \in \mathbb{H}$, which is

$$\overline{z} = a_0 - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}.$$

Here, $\mathrm{Re}(z) = a_0$ and $\mathrm{Im}(z) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

Remark Note that $z = \overline{z} \iff z \in \mathbb{R}$ and $z = -\overline{z} \iff z$ is pure imaginary.

Lemma

If $z = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \in \mathbb{H}$ *where* $a_j \in \mathbb{R}$, *then*

$$z\overline{z} = \overline{z}z = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Proof. We have

$$\begin{aligned}
z\overline{z} &= (a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(a_0 - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}) \\
&= a_0^2 - a_0a_1\vec{i} - a_0a_2\vec{j} - a_0a_3\vec{k} \\
&+ a_1^2 + a_0a_1\vec{i} + a_1a_3\vec{j} - a_1a_2\vec{k} \\
&+ a_2^2 - a_2a_3\vec{i} + a_0a_2\vec{j} + a_1a_2\vec{k} \\
&+ a_3^2 + a_2a_3\vec{i} - a_1a_3\vec{j} + a_0a_3\vec{k} \\
&= a_0^2 + a_1^2 + a_2^2 + a_3^2.
\end{aligned}$$

In fact,

$$z\bar{z} = \det \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},$$

with $z_1 = a_0 + a_1 i, z_2 = a_2 + a_3 i$.The proof for the other formula is similar. $\square$

We define the **norm or absolute value** of $z \in \mathbb{H}$ as $|z| = \sqrt{z\bar{z}} = |\bar{z}|$. Here are some properties of quaternions.

Theorem

Let $z, z_1, z_2 \in \mathbb{H}$.

(1). $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$;

(2). $\overline{\overline{z_1}} = z_1$;

(3). $\overline{z_1 z_2} = \overline{z}_2 \overline{z}_1$;

(4). $z_1 \overline{z}_1 = \overline{z}_1 z_1 \geq 0$ *and* $z_1 \overline{z}_1 = 0 \iff z_1 = 0$;

(5). $Z(\mathbb{H}) = \mathbb{R}$, *i.e., the centre of* $\mathbb{H}$ *is* $\mathbb{R}$;

(6). $|z_1 z_2| = |z_1| \cdot |z_2|$;

(7). *If* $z \neq 0$, *then* z *is invertible and* $z^{-1} = \dfrac{\overline{z}}{|z|^2}$;

(8). $\overline{z_1^{-1}} = (\overline{z}_1)^{-1}$ *if* $z_1 \neq 0$. *Thus* $\mathbb{H}$ *is a division ring.*

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(5). Suppose that $z = a_0 + a_1 i + a_2 j + a_3 k \in Z(\mathbb{H})$. From $iz = zi$ we have $a_0 i - a_1 + a_2 k - a_3 j = a_0 i - a_1 - a_2 k + a_3 j$, yielding that $a_2 = a_3 = 0$. Then from $zj = jz$ we deduce that $a_1 = 0$. Thus $z \in \mathbb{R}$, i.e., $Z(\mathbb{H}) \subset \mathbb{R}$. It is clear that $\mathbb{R} \subset Z(\mathbb{H})$. So $Z(\mathbb{H}) = \mathbb{R}$.

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(6). We see that $|z_1 z_2| = |z_1| \cdot |z_2|$:

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$$\begin{aligned} z \cdot \overline{z} = \overline{z} z = |z|^2 > 0 &\iff |z|^{-2} \cdot (z \cdot \overline{z}) = |z|^{-2} \cdot (\overline{z} \cdot z) = 1 \\ &\iff z \cdot (|z|^{-2} \cdot \overline{z}) = (|z|^{-2} \cdot \overline{z}) \cdot z = 1 \\ &\iff z^{-1} = \frac{\overline{z}}{|z|^2}; \end{aligned}$$

Note that z^{-1} is indeed the inverse matrix of the matrix z.

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We know that $axb = c \iff a^{-1}axbb^{-1} = a^{-1}cb^{-1}$. So $x = a^{-1}cb^{-1}$. However, one cannot easily solve the quadratic equation $x^2 + axb + c = 0$, where $a, b, c \in \mathbb{H}$. Actually there is no formula like the quadratic formula over $\mathbb{C}$. There are a lot of studies on this now!

Lemma

Let $a, b \in \mathbb{H}$.

(1). $ab = ba \iff \mathrm{Im}(a)$ *and* $\mathrm{Im}(b)$ *are linearly dependent.*

(2). $a^2 = -1 \iff |a| = 1$ *and* $\mathrm{Re}(a) = 0$.

Recall the definition for two matrices to be similar. In $\mathbb{H}$, two quaternions $g_1, g_2 \in \mathbb{H}$ are **similar** if there exists $h \in \mathbb{H}$ such that $g_2 = hg_1h^{-1}$, denoted by $g_1 \sim g_2$.

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It is easy to see that the similarity in $\mathbb{H}$ is an equivalence relation on $\mathbb{H}$.

Theorem

Let $a, b \in \mathbb{H}$. Then $a \sim b$ if and only if $\mathrm{Re}(a) = \mathrm{Re}(b)$ and $|\mathrm{Im}(a)| = |\mathrm{Im}(b)|$.

Proof. We write $a = a_0 + a', b = b_0 + b'$ as before where $\mathrm{Re}(a) = a_0, \mathrm{Re}(b) = b_0$.
$(\Rightarrow)$. Suppose that $a \sim b$. Then $b = xax^{-1}$ for some $x \in \mathbb{H}$. Then

$$\begin{aligned} b_0 + b' = x(a_0 + a')x^{-1} &= xa_0x^{-1} + xa'x^{-1} \\ = a_0 + xa'x^{-1} &= a_0 + xa'\frac{\overline{x}}{|x|^2} \end{aligned}$$

and note that

$$\begin{aligned} \overline{xa'x^{-1}} = \frac{\overline{xa'\overline{x}}}{|x|^2} &= |x|^{-2}\overline{\overline{x}}\,\overline{a'}\,\overline{x} \\ = -|x|^2 xa'\overline{x} &= -xa'x^{-1} \end{aligned}$$

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so it must follow that $xa'x^{-1}$ is pure imaginary. We see that $a_0 = b_0, b' = xa'x^{-1}$. Therefore $a_0 = b_0, |b'| = |a'|$.

$(\Leftarrow)$. $a_0 = b_0, |a'| = |b'|$. Therefore we have

$$-a'a' = a'\overline{a'} = |a'|^2 = |b'|^2 = -b'b',$$

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Case 1: $a' + b' \neq 0$.
Take $x = b' + a'$. We need to show that $xax^{-1} = b$, i.e., $xa = bx$:

$$(a' + b')(a_0 + a') = (a_0 + b')(a' + b')$$

which reduces to

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Case 2: $a' + b' = 0$.

We can easily take a pure imaginary c' such that $c' \neq \pm a'$ and $|c'| = |a'| = |b'|$. Using Case 1 we deduce that $a \sim a_0 + c' \sim b$. □

We will have the following theorem with providing the proof.

Theorem (Wedderburn's Theorem)

Every finite division ring is a field.

Polynomial Equations Over $\mathbb{H}$

Unlike complex numbers:

- Left zeros $\neq$ right zeros.
- Factorization is subtle.
- No unique factorization.

Eilenberg–Niven (1944):
Every quaternionic polynomial with a single highest-degree term has at least one zero.

Beginning of systematic theory.

Linear Quaternionic Equations

Simple cases:

$$ax = b, \quad xa = b$$

Solvable because $\mathbb{H}$ is division algebra.

More difficult:

$$ax + xb = c.$$

Helmstetter (2012): Classification of solutions in degenerate cases.

[C. Shao, H. Li, L. Huang, Basis-free solution to general linear quaternionic equation, *Linear and Multilinear Algebra*, 68 (2020), 435–457.]

General Two-Sided Linear Equation

General form:

$$f(x) = \sum_{m=1}^{n} a_m x b_m.$$

Noncommutativity prevents:

- Standard matrix representation over $\mathbb{H}$
- Direct determinant methods

Previous methods:

- Rearrangement techniques
- Fixed-point theorems
- Basis-free algebraic manipulations

But no unified linear-algebraic framework.

Viewing $\mathbb{H}$ as $\mathbb{R}^4$

Key observation:

$$\mathbb{H} \cong \mathbb{R}^4$$

Basis:

$$\{1, i, j, k\}.$$

Any $x \in \mathbb{H}$:

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

Thus:

$$f(x) \text{ is } \mathbb{R}\text{-linear.}$$

[Y. Chen, B. Liu, B. Wang, K. Zhao, Two-sided Linear Quaternionic Equations, *Journal of Algebra and its Applications*, https://doi.org/10.1142/S0219498826400013.]

Definition: Discriminant Space

Let

$$f(x) = \sum_{m=1}^{n} a_m x b_m.$$

Define the **discriminant space** associated with $f(x)$ as:

$$V_f = \mathrm{Span}_{\mathbb{R}}\{f(1), f(i), f(j), f(k)\}.$$

Index of $f(x)$:

$$I_f = 4 - \dim_{\mathbb{R}}(V_f).$$

Interpretation:
V_f = image of f as a real-linear transformation.

Main Theorem (Single Equation)

Consider

$$f(x) = c.$$

Theorem

1. Solvable $\iff$ $c \in V_f$.
2. If solvable:
$$\dim(\text{solution space}) = I_f.$$
3. Unique solution $\iff$ $I_f = 0$.

Proof Idea

Write:

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

Then:

$$f(x) = x_0 f(1) + x_1 f(i) + x_2 f(j) + x_3 f(k).$$

Thus:

$$c \in \mathrm{Span}_{\mathbb{R}}\{f(1), f(i), f(j), f(k)\}.$$

Matrix Formulation

Construct matrix:

$$A = \begin{pmatrix} | & | & | & | \\ f(1) & f(i) & f(j) & f(k) \\ | & | & | & | \end{pmatrix}$$

So the problem reduces to solve such a real linear system

$$\mathbf{A} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Matrix Formulation

Interpreted in $\mathbb{R}^4$.

Then:

$$\operatorname{rank}(A) = \dim_{\mathbb{R}}(V_f).$$

Solution dimension:

$$4 - \operatorname{rank}(A).$$

Example 1

Solve:

$$x + ixj + jxi = i.$$

Let $f(x) = x + \mathbf{i}x\mathbf{j} + \mathbf{j}x\mathbf{i}$. Then

$$f(1) = 1, \quad f(\mathbf{i}) = \mathbf{i} - 2\mathbf{j}, \quad f(\mathbf{j}) = -2\mathbf{i} + \mathbf{j}, \quad f(\mathbf{k}) = \mathbf{k}.$$

From the Theorem, we deduce that th above has a unique solution since $\mathbf{i} = -\frac{f(\mathbf{i})+2f(\mathbf{j})}{3} \in V_f$ and $I_f = 0$. Note that

$$\begin{bmatrix} f(1) & f(\mathbf{i}) & f(\mathbf{j}) & f(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix}.$$

Thus the unique solution is

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = -\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j}.$$

Example 2: One-Dimensional Solution Space

Equation:

$$x + (i + j)xj + (i + j)xi - kxk = 2i + 3j + 5k.$$

Compute:

$$\dim V_f = 3.$$

Thus:

$$I_f = 1.$$

Solution space is one-dimensional affine subspace.

General System

$$f_i(x_1, x_2, \cdots, x_n) = \sum_p a_{p,1}^{(i)} x_1 b_{p,1}^{(i)} + \sum_p a_{p,2}^{(i)} x_2 b_{p,2}^{(i)} + \cdots + \sum_p a_{p,n}^{(i)} x_n b_{p,n}^{(i)},$$

where $a_{p,j}^{(i)}, b_{p,j}^{(i)} \in \mathbb{H}$, for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.
Let $e_1 = (1, 0, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_n = (0, 0, \cdots, 1)$ be the standard basis vectors in $\mathbb{Z}^n$.

$$f_i(x_1, \ldots, x_n) = c_i.$$

Each $c_j \in \mathbb{H}$.
Unknown dimension (real): $4n$
Define:

$$V_F = \mathrm{Span}_{\mathbb{R}}\{u_{j,r}\},$$

where

$$u_{j,0} = (f_1(e_j), f_2(e_j), \cdots, f_m(e_j)), \tag{1}$$
$$u_{j,1} = (f_1(e_j\mathbf{i}), f_2(e_j\mathbf{i}), \cdots, f_m(e_j\mathbf{i})), \tag{2}$$
$$u_{j,2} = (f_1(e_j\mathbf{j}), f_2(e_j\mathbf{j}), \cdots, f_m(e_j\mathbf{j})), \tag{3}$$
$$u_{j,3} = (f_1(e_j\mathbf{k}), f_2(e_j\mathbf{k}), \cdots, f_m(e_j\mathbf{k})). \tag{4}$$

Index: $I_F = 4n - \dim V_F$.

Main Theorem (System)

1. Solvable $\iff (c_1, \ldots, c_m) \in V_F$.
2. dim(solution space) $= l_F$.
3. Unique solution $\iff l_F = 0$.

Again reduces to:

$$A\boldsymbol{x} = \boldsymbol{c}$$

over $\mathbb{R}$.

Conjugate Identity

Key identity:

$$\bar{x} = -\frac{1}{2}(x + ixi + jxj + kxk).$$

Remark:
No analogue in complex numbers.

Reduction Strategy

Equation:

$$f(x, \bar{x}) = \sum a_i x b_i + \sum c_j \bar{x} d_j.$$

Substitute conjugate identity.
Obtain:

$$f(x, \bar{x}) = \sum a_i^* x b_i^*.$$

Reduce to previous case.

Comparison with Previous Approaches

Previous methods:

- Fixed-point arguments
- Rearrangement identities
- Clifford algebra techniques

This paper:

- Pure linear algebra
- Rank criterion
- Dimension formula
- Unified treatment

Geometric Picture

Think of:

$$f : \mathbb{R}^4 \to \mathbb{R}^4.$$

Then:

- V_f = image
- I_f = nullity
- Rank-nullity theorem applies

Noncommutativity disappears at the real level.

Extensions

- Octonions (non-associative case)
- Quaternionic matrices
- Polynomial systems
- Numerical algorithms

Potential applications:

- Quantum mechanics
- Robotics (dual quaternions)
- Signal processing

Open Questions

How do we solve two-sided quaternionic quadratic equations?

[B. Kalantari, Algorithms for quaternion polynomial root-finding, *J. Complexity*, 29 (2013), 302–322.]

[L. Feng, K. Zhao, Classifying zeros of two-sided quaternionic polynomials and computing zeros of two-sided polynomials with complex coefficients, *Pacific J. Math.*, 262 (2013), 317–337.]

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Conclusion

Main insight:

Two-sided quaternionic linear equations

$$\Downarrow$$

Real-linear algebra in dimension 4.

Discriminant space gives:

- Solvability criterion
- Dimension formula
- Unified framework

References

Hamilton (1843)
Eilenberg–Niven (1944)
Helmstetter (2012)
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Thank You!